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Large deviation statistics of the energy-flux fluctuation in the shell model of turbulence

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The energy-flux fluctuation in the shell model of turbulence is numerically analyzed from the large deviation statistical point of view. We first observe that the rate function defined in the inertial range is independent of the Reynolds number. The rate function derived by the cascade model of the log-Poisson statistics turns out to be in good agreement with the present numerical result in the region where strong singularity of fluctuation exits. This fact may imply the universality as well as the robustness of the large deviation statistical quantities in turbulence.

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The physical process of fully developed turbulence (Reynolds number $\mathrm{Re}\gg 1$) is well established as the energy transfer process from the energy injection scale L to the dissipation scale η. The statistical property of turbulence in the inertial range scale $l(\eta\ll l\ll L)$ will be universal, which is mainly measured by the two quantities, the longitudinal velocity difference δv_l across a distance l and the energy dissipation ϵ_l averaged over a region of scale l. Much of the experimental and theoretical research are concerned with the scaling exponents $\zeta(q)$ and $\tau(q)$ which characterize the scaling behaviors of the structure functions defined as

$$\langle|\delta v_l|^q\rangle\sim V_0^q\left(\frac{l}{L}\right)^{\zeta(q)},\quad \langle\epsilon_l{}^q\rangle\sim\epsilon_0^q\left(\frac{l}{L}\right)^{\tau(q)},\tag{1}$$

where $\langle\cdots\rangle$ being the ensemble average, V_0 and $\epsilon_0(=\epsilon_L)$ represent the characteristic velocity and energy dissipation rate with scale L, respectively. By supposing that ϵ_l is spatially uniform provided that the scale l is in the inertial range, the relation $\zeta(q)=q/3$ and $\tau(q)=0$ are derived by Kolmogorov (K41) [1]. However, it is widely known from experiments or direct numerical simulations that the q dependences of $\zeta(q)$ and $\tau(q)$ are different from the K41 law. This is noticed as the intermittency problem, and the strong fluctuations of the velocity field or energy dissipation field have been discussed in connection with the intermittent cascade dynamics [2].

The Kolmogorov 1962 theory (K62) [3], the first theoretical approach to the intermittency problem, discussed the statistics of ϵ_l in a concrete manner. The key idea is the self-similarity hypothesis on fluctuations of energy dissipation. Namely, by introducing the ratio of $\epsilon_{l_n}(=\epsilon_n)$ averaged over the scale $l_n=\lambda^{-n}L$ $(\lambda>1)$ and that over the scale l_{n+1},

$$\frac{\epsilon_{n+1}}{\epsilon_n}=\exp_\lambda(z_n),\tag{2}$$

where $\exp_\lambda(\theta)\equiv\lambda^\theta$, the statistical property of z_n is supposed to be random, and is free from the scale level n. One should note that Eq. (2) holds for $n=0,1,\ldots,\log_\lambda(L/\eta)(\equiv N)$. Let us introduce the local average value of z_j by $\overline{z_n}=\Sigma_{j=0}^{n-1}z_j/n=(\log_\lambda\epsilon_n-\log_\lambda\epsilon_0)/n$, where $\epsilon_0=\overline{\epsilon}$ is assumed to be constant. If the probability distribution function (PDF) of z_j is supposed to be identical and free from j, PDF for $\overline{z_n}$, $Q_n(z)=\langle\delta(\overline{z_n}-z)\rangle$ may obey the normal distribution for a large n because of the central limit theorem (CLT). Namely, we suppose that PDF of ϵ_n obeys the log-normal distribution. This yields the intermittency exponent as $\tau(q)=-\mu q(q-1)/2$ $[\mu\equiv-\tau(2)]$ and $\zeta(q)=q/3-\mu q(q-3)/18$ is obtained by using the refined similarity hypothesis $\delta v_l\sim l^{1/3}\epsilon_l{}^{1/3}$.

The result of K62 is thus characterized by the quadratic curves of $\zeta(q)$ and $\tau(q)$. In fact, K62 is considered to be a good approximation for low order moments but not for high order ones. This is because the Gaussian characteristics of $Q_n(z)$ are limited to the weak fluctuation region, i.e., $|\overline{z_n}-z^*|\sim O(1/\sqrt{n})$ with the ensemble average z^* of z_n. Particularly, the statistics of the strong singularity of fluctuation cannot be described by the K62 approximation. In such a case, instead of CLT, one can discuss the nature of PDF by utilizing the large deviation theory (LDT), which is a generalization of CLT [2,4]. LDT insists that PDF of $\overline{z_n}$ asymptotically takes the form

$$Q_n(z)\propto\sqrt{n}\exp_\lambda[-S(z)n]\tag{3}$$

for $n\gg n_c$, n_c being the correlation step of z_n. The function $S(z)$, being independent of n, is called the rate function [2] or the fluctuation spectrum [4] and characterizes the asymptotic form of PDF. Ergodicity of z_n requires that $S(z)$ is a concave function and takes a minimum value 0 at z^*. Therefore, PDF for $\epsilon_n=\epsilon_0\exp_\lambda(n\overline{z_n})$ is written as $P_n(\epsilon)$

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$\sim\exp_\lambda[-nS(n^{-1}\log_\lambda(\epsilon/\epsilon_0))]/(\sqrt{n}\epsilon)$. Since PDF of ϵ_n for small scale l is asymptotically characterized by $S(z)$, $S(z)$ plays the central role of the fluctuation statistics of ϵ_n. Moreover, the moments defined as $\langle \epsilon_n{}^q\rangle = \epsilon_0{}^q \langle \exp_\lambda(nq\overline{z_n})\rangle$ lead to the characteristic function $\tau(q)$ via $\langle \exp_\lambda(nq\overline{z_n})\rangle \propto \exp_\lambda[-n\tau(q)]$ for large n. Using Eq. (3), we get

$$\langle \epsilon_n{}^q \rangle \propto \int \exp_\lambda[-n\{S(z)-qz\}]dz. \tag{4}$$

The above integral is evaluated by the steepest descent method for large n by supposing $S''(z)>0$. We obtain

$$\tau(q)=\min_{\acute{z}}[S(\acute{z})-q\acute{z}], \tag{5}$$

which is identical to the Legendre transform between $\{z,S(z)\}$ and $\{q,\tau(q)\}$. Since $S(\acute{z})-q\acute{z}$ has the unique minimum at $\acute{z}=z(q)$, Eq. (5) yields $\tau(q)=S(z(q))-qz(q)$, $dS(z)/dz=q$, $z(q)=-\tau'(q)$ and $\chi(q)=z'(q)=-\tau''(q)$. From the data analysis point of view, when we can get a large amount of data ensemble, the statistical analysis by PDF of ϵ_n may be better than calculating all the moments of ϵ_n. Small z describes a weak intermittency (laminarlike) statistics, and a strong intermittency (burst) is characterized by large z-values. Therefore, the form of $S(z)$ is directly related to the realization probability of various intermittent features of turbulent field.

When the local scaling exponent α is defined as $\epsilon_l/\epsilon_L \sim (l/L)^{\alpha-1}$ to discuss the multifractality of the fluctuation of energy dissipation field, the Hausdorff dimension $f(\alpha)$ of the support with the value α is related to $S(z)$ as [2]

$$S(z)=3-f(1-z). \tag{6}$$

The treatment of intermittency from the present LDT viewpoint is thus essentially consistent with the multifractal formalism of turbulence [2,5]. One should note that the concept of the multifractal is not clearly used in the present approach. Namely, it should be stressed that the function $S(z)$ can be defined as a self-similarity quantity without connection to the multifractal structure in real turbulent field. In the present Rapid Communication, to analyze the intermittency statistics, we directly calculate $S(z)$ by numerically integrating the shell model of turbulence defined by the physical quantities in the wave number space, and compare the result with theories.

There are dynamical models based on the energy cascade picture to understand the turbulence from the dynamical system viewpoint [6]. We take the so-called (GOY) shell model [7], which is the dynamical system composed of $2N$ dimensional differential equations

$$\frac{du_n}{dt}=iC_n-\nu k_n{}^2 u_n+\delta_{n,4}f, \tag{7}$$

where $C_n=k_n u^*_{n+1}u^*_{n+2}-k_{n-1}u^*_{n-1}u^*_{n+1}/2 - k_{n-2}u^*_{n-1}u^*_{n-2}/2$ is the nonlinear term and $k_n=2^n k_0$ ($n=1,\ldots,N$) corresponds to the wave number of the nth shell. u_n is the complex variable assigned to the shell number n. These equations are regarded as the reduction model of the Navier-Stokes equation (NS). One of the important differences between NS and GOY is the nonlinear terms modeled by the local interaction of u_n in the wave number space. The intermittent properties of GOY have been extensively discussed, e.g., in [8,9] and the scaling exponent $\zeta(q)$ defined by $\langle |u_n|^q\rangle \sim k_n^{-\zeta(q)}$ in the inertial range turns out to be anomalous, i.e., different from the K41 law. This fact is consistent with the result in real turbulence. Physically, the fluctuation of energy-flux function

$$F_n(t)=-k_n\,\mathrm{Im}\left(u_n u_{n+1}u_{n+2}+\frac{1}{4}u_{n-1}u_n u_{n+1}\right) \tag{8}$$

is more relevant to the intermittency dynamics rather than u_n itself. In the inertial range scale, we expect that $\langle |F_n|^q\rangle \sim k_n^{-\tau(q)}$ and $\zeta(q)=q/3+\tau(q/3)$ is established because of the scaling relation $|F_n|\sim k_n|u_n|^3$. Moreover, $|F_n|$ is a physical quantity directly characterizing the cascade dynamics in the inertial range rather than the energy dissipation rate. So we investigate the fluctuation property of $|F_n|$ by calculating $S(z)$.

In GOY, ϵ_0 corresponds to the temporal average of energy dissipation rate $\epsilon(t)=\nu\Sigma_{i=1}^N k_i^2|u_i|^2$, which is equivalent to $\langle F_n\rangle$ in the inertial range. We take $\langle |F_n|\rangle$ instead of $\langle F_n\rangle$ to keep the equality $\tau(1)=0$, and calculate PDF $Q_n(z)$ for $\overline{z_n}=(\log_2|F_n|-\log_2\langle|F_n|\rangle)/n$ to evaluate $S(z)$ by

$$S(z)\simeq -\frac{1}{n}\log_2 Q_n(z). \tag{9}$$

For large n, i.e., in deeper shells in the inertial range, it is expected that $S(z)$ obtained from Eq. (9) is sufficiently converged [10]. In the present study, we take 10^7 data points to construct PDF from the numerical simulation of GOY. The parameters are chosen as $k_0=2^{-4}$ and $f=5.0(1+i)\times 10^{-3}$ with the total shell number $N=22$, $\nu=10^{-7}$ (RUN1) and $N=27$, $\nu=10^{-9}$ (RUN2). For numerical integration scheme, the second order Adams-Bashforth scheme [6] with the time increment of $\Delta t=5.0\times 10^{-5}$ is used. Under this condition, the inertial range, which is defined as that where long-time average of F_n does not practically depend on n, is in $n=5-15$ for RUN1 and $n=7-20$ for RUN2. Generally $S(z)$ obtained via Eq. (9) is not in agreement with $S(z(q))$ obtained via Legendre transform of $\tau(q)$ due to the finite size effect of n. So $S(z)$ evaluated from $Q_n(z)$ are slightly shifted along the abscissa so that the minimum position of $S(z)$ coincides with $z^*=\langle\overline{z_n}\rangle$ for large n, where z^* is numerically determined by assuming $\langle\log_\lambda|F_n|\rangle = z^*n+C$, C being a constant, in the inertial range. This prescription is not crucial if we are interested in the functional form of $S(z)$ converged in the finite inertial range n.

The temporal evolutions of $F_n(t)$ in the inertial range shells $n=12$, 13, and 14 for RUN1 are shown in Fig. 1. The dynamics of F_n represents intermittent characteristics in the process of the energy transfer from a large scale motion to small one, and consists of the two phases, laminar and burst phases. In addition, we observe that the intensity of intermittent fluctuation is strong in high wave number shells more than low wave number ones, which may cause the self-similar statistical nature of F_n with respect to shell variables with different shell numbers. Figure 2 represents the rate

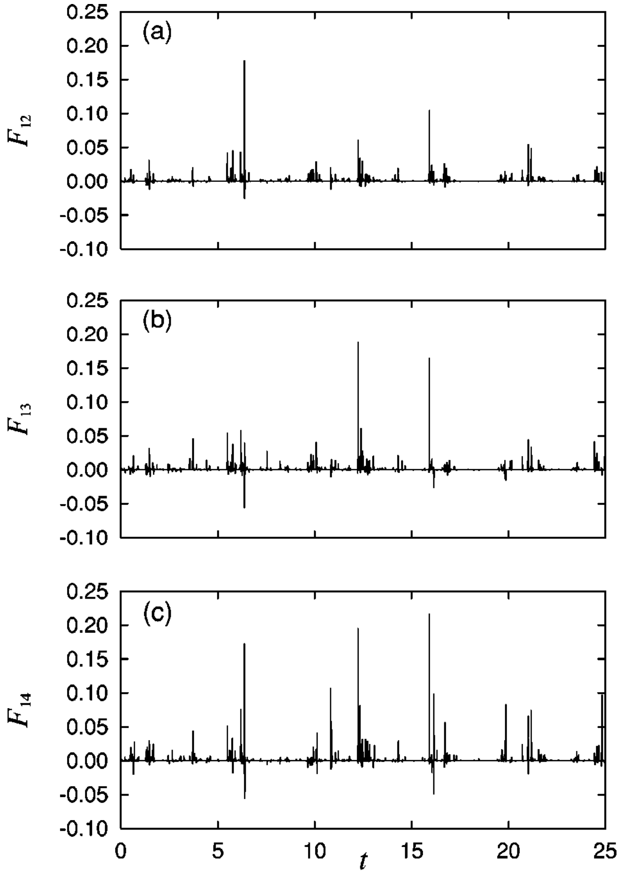

FIG. 1. Time evolutions of $F_n(t)$ in the inertial range shells (a) $n=12$, (b) 13, and (c) 14 for RUN1.

function obtained by measuring $Q_n(z)$ for energy-flux fluctuation in Fig. 1. One clearly finds that each rate function in the inertial range shell is on the same curve, which may be the universal function characterizing the intermittent energy

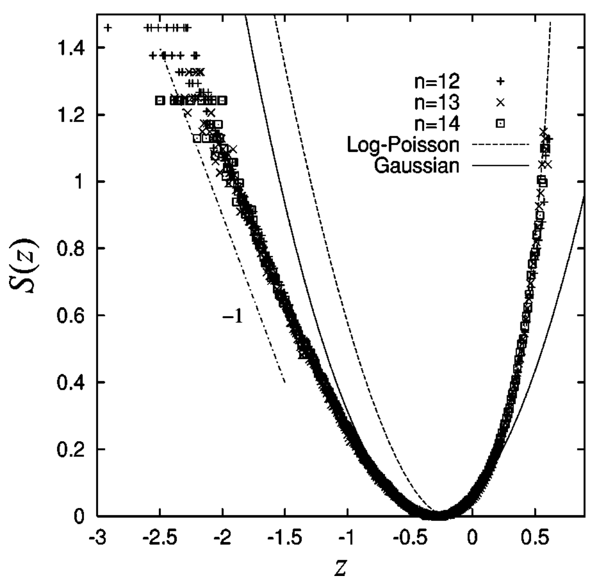

FIG. 2. Rate function calculated from Eq. (9) for the data of Fig. 1. Dotted line represents the rate function derived by the She-Leveque model [Eq. (11)] and the straight line represents the quadratic curve around the minimum of data [the Gaussian approximation for $Q_n(z)$].

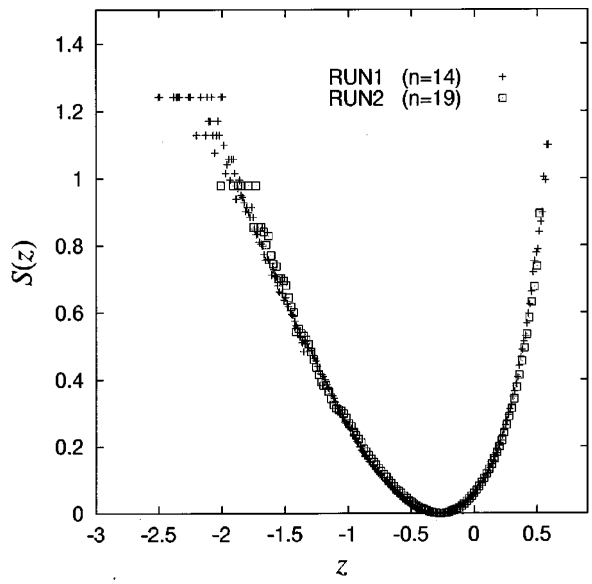

FIG. 3. Rate functions converged in the inertial range for shell numbers $n=14,19$ for RUN1, 2, respectively.

cascade dynamics. Moreover we must notice that $S(z)$ is a concave function in a wide region. Left edge of $S(z)$ approximately takes the form $S(z)=-a(z-z_m)$ with a positive constant a and a constant z_m. In this region, PDF of $|F_n|$ is represented as $P(|F_n|)\sim|F_n|^{a-1}$. Numerical result shows $a=1$, which implies that PDF is finite at $|F_n|=0$. This may originate from the inverse energy cascade process. This nature is quite different from the statistics of the energy dissipation rate, which is a positive defined variable, and reflects the characteristics of fluctuation of energy-flux fluctuation. A great interest on the inertial range statistics is the Reynolds number dependence. The converged rate functions in the inertial range are obtained for two different Re runs (RUN1 and RUN2). We use the data of the shell numbers $n=14$ and 19 for RUN1 and 2, respectively. Figure 3 clearly shows the converged rate functions are independent of Re.

Noting the concavity $S''(z)>0$ guaranteed from the numerical results, the rate function around its minimum can be approximated as

$$S(z)\approx\frac{1}{2\chi(0)}[z-z(0)]^2. \quad (10)$$

This approximation is limited in the region where CLT can be applied. But if we apply the form of Eq. (10) to all the regions of z, the relation $z(0)=-\chi(0)/2$ is required, which is identical to the K62 theory. In this case, the intermittency exponent is defined by $\mu=\chi(0)=-\tau(2)$ and the quadratic curve is determined by the one parameter μ. Let us give a comment on the applicability of the asymptotic form of Eq. (10). Although $z(0)$ and $\chi(0)$ have no interrelation in general, the K62 theory assumes the relation $z(0)=-\chi(0)/2=\tau(2)/2$. This implies that K62 is fitted around the neighborhood of $z=z(2)$, not $z(0)$ by defining the intermittency exponent $\mu\equiv-\tau(2)$. The present numerical study gives $z^*=z(0)=-0.31$ and $-\chi(0)/2=-1/[2S''(z^*)]=-0.38$ from Fig. 2, which are clearly different from the prediction of K62. The quadratic curve obtained from the Gaussian approximation around the minimum is shown in Fig. 2. As a natural result, the Gaussian approximation is valid around $z = z(0)$ and the fluctuations are characterized by the form of $S(z)$ which is generally different from the quadratic curve except the neighborhood of the minimum.

If one discusses the statistical law for intermittency in fully developed turbulence from the LDT viewpoint, the following two questions arise: (i) Are $z(0)$ and $\chi(0)$ really constants in the high Reynolds number limit? and (ii) how does $S(z)$ deviate from the quadratic curve predicted by CLT? The first question is connected to the Re dependence of applied region of CLT for $S(z)$. Experimental results deviate from the log-normal distribution as Re is increased [11]. If $S(z)$ is well-defined for $\mathrm{Re}\to\infty$, $z(0)$ and $\chi(0)$ must be constants. The exponent μ, well known as a universal constant characterizing intermittency, has been investigated in detail, e.g., in [11], but there is no particular reason why all statistics are determined only by one parameter μ. We insist that $z(0)$ and $\chi(0)$ are important quantities to investigate universal features of intermittency in turbulence. This point has not been pointed out so far. The second problem is related to the statistical nature of strong or weak singularities of fluctuations. Various phenomenological cascade models are constructed for $\zeta(q)$ or $\tau(q)$ to explain the experimental or numerical simulation results [2]. In this paper, the rate function $S(z)$ of GOY is compared with that obtained from the Legendre transformation of the She-Leveque model (SL) derived by applying the Log-Poisson statistics as the multiplicative cascade [12]. The SL yields $\tau(q) = -\gamma q + \gamma(1-\beta^q)/(1-\beta)$ [13], which explains the results of experiments or intermittent scaling of shell models excellently. This is equivalent to

$$S(z) = \frac{z-\gamma}{\log\beta}\left[\log\left(\frac{(z-\gamma)(1-\beta)}{\gamma\log\beta}\right)-1\right]+\frac{\gamma}{1-\beta}. \quad (11)$$

Two parameters, γ and β, are chosen as $\gamma=0.625$ and $\beta=0.58$, which are numerically estimated from $\zeta(q)$ of GOY in [9]. The comparison with the present experimental results is made in Fig. 2. The present result and SL are in good agreement with each other in the right region of $S(z)$, i.e., in the burst dominant region. Characteristics of strong singularity of fluctuation are thus excellently explained by SL. On the other hand, the left region of $S(z)$ are extremely different from SL, which implies that the weak singularity of fluctuations cannot be described by SL. The feature of SL pointed out above agrees with that from the direct numerical simulation of NS [14] or the data analysis of real turbulent flow [15].

In this Rapid Communication, we discussed the statistical nature of intermittency of energy-flux fluctuation with the GOY shell model from the LDT point of view by calculating the rate function $S(z)$ from PDF. We found that $S(z)$ in the inertial range is independent of Re, and that $S(z)$ for the z region reflecting the strong singularity of fluctuation agrees quite well with SL. This will support the result of [9] indicating $\zeta(q)$ of GOY is in good agreement with SL. However, SL cannot describe the correct statistics of GOY in a weak singularity region. K62 explains the weak fluctuation statistics around the minimum of $S(z)$ and the strong fluctuation is described by SL quite well. It is quite natural that each cascade model usually has its own applicability. It is surely a great challenge to construct a unified cascade model explaining all the statistical properties of energy-flux fluctuations.

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